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A Note on Occupation Times of Random Walks

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In this note I show that some asymptotic results on the average occupancy time of an interval derived for lattice random walks with negative exponential transition probabilities are true for all random walks whose transition probabilities have a finite variance. The proof is based on the continuum limit.

KEY WORDS: Lattice random walks; occupation times; continuum limit; first passage times.

In a recent paper Gutkowitz-Krusin *et al.*[1] have derived results for the occupation times and adsorption probabilities of a one-dimensional random walk with exponential transition probabilities. They did so by exact calculations that were possible because of the particular form of transition probabilities. In this note we point out that the asymptotic results derived by these investigators do not depend on the specific transition probabilities but only on the fact that the variance of the single-step jump probabilities is finite.

Let the single-step jump probabilities be denoted by p_j, i.e., p_j is the probability that the random walker takes a step of size j at a given transition time. We define the structure factor to be $\lambda(\theta)$, where

$$\lambda(\theta) = \sum_{j=-\infty}^{\infty} p_j e^{ij\theta} \tag{1}$$

and let the lth moment be

$$\mu_l = \sum_{j=-\infty}^{\infty} j^l p_j \tag{2}$$

In what follows we will assume that $|\mu_1| < \infty$ and that

$$\sigma^2 = \mu_2 - \mu_1^2 < \infty \tag{3}$$

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Both of these requirements are trivially satisfied by the exponential transition model. The model of Ref. 1 assumes that the infinite line can be divided into three segments, $(-\infty, -1)$, $(0, N)$, $(N+1, \infty)$, such that the two outer segments consist only of traps. One is then required to find the mean occupation time of the segment $(0, N)$ as well as the probability of being trapped in the right or left segments, respectively. We will work in the asymptotic limits $N \gg \mu_1$, $N\mu_1 \gg \sigma^2$.

Let $Q_n(r|r_0)$ be the probability that at step n the random walker is at r given that $Q_0(r|r_0) = \delta_{r,r_0}$. The $Q_n(r|r_0)$ satisfy the master equation

$$Q_n(r|r_0) = \sum_{\rho=-\infty}^{\infty} Q_{n-1}(\rho|r_0)p_{r-\rho} \tag{4}$$

In the limit $N \gg \mu_1$ we can approximate the master equation by a partial differential equation

$$\frac{\partial Q_n}{\partial n} = \frac{\sigma^2}{2}\frac{\partial^2 Q_n}{\partial r^2} - \mu_1 \frac{\partial Q_n}{\partial r} \tag{5}$$

In this approximation the equation just written is to be solved subject to the initial and boundary conditions

$$Q_0(r|r_0) = \delta(r - r_0), \qquad Q_n(0|r_0) = Q_n(N|r_0) = 0 \tag{6}$$

We first consider the problem of finding the mean residence time in $(0, N)$ on the assumption that the initial position is uniformly distributed throughout the interval. The probability that absorption has not occurred by step n is

$$P_n = (1/N) \int_0^N dr \int_0^N dr_0 \, Q_n(r|r_0) \tag{7}$$

and the mean residence time is just

$$\langle n \rangle = (1/N) \int_0^N dr \int_0^N dr_0 \int_0^\infty dn \, Q_n(r|r_0) \tag{8}$$

One can use standard methods to show that the solution to Eq. (5) subject to Eq. (6) is

$$Q_n(r|r_0) = \frac{2}{N}\sum_{j=0}^{\infty} \exp\left\{-\frac{\pi^2 j^2 \sigma^2 n}{2N^2} - \frac{\mu^2 n}{2\sigma^2} + \frac{\mu}{\sigma^2}(r - r_0)\right\} \sin\frac{\pi j r}{N} \sin\frac{\pi j r_0}{N} \tag{9}$$

An alternative expression can be found by making a Poisson transformation of this last expression, which leads to the result

$$\begin{aligned} Q_n(r|r_0) = {} & \frac{1}{(2\pi n\sigma^2)^{1/2}} \exp\left\{\frac{1}{2n\sigma^2}[2n\mu(r - r_0) - \mu^2 n^2]\right\} \\ & \times \sum_{l=-\infty}^{\infty} \left\{\exp\left[-\frac{1}{2n\sigma^2}(2lN + r - r_0)^2\right] \right. \\ & \left. - \exp\left[-\frac{1}{2n\sigma^2}(2lN + r + r_0)^2\right]\right\} \end{aligned} \tag{10}$$

When $\mu_1 \neq 0$ the lowest order approximation to n can be found by choosing only the $l = 0$ term, leading to

$$\langle n \rangle \sim \frac{N}{2\mu_1} - \frac{\sigma^4}{4N\mu_1{}^3} + O\left(e^{-2N\mu_1/\sigma^2}\right) \tag{11}$$

Thus, to a first approximation, when $\mu_1 \neq 0$, the effect of diffusion can be neglected. When $\mu_1 = 0$ another result is obtained by substituting the series in Eq. (9) into the integral of Eq. (8). This calculation leads to the result

$$\langle n \rangle \sim N^2/(6\sigma^2) \tag{12}$$

The leading term of Eq. (11) and this last result are identical to the results given in Ref. 1 but have a wider application since they do not depend on the particular form of transition probabilities.

One can develop a similar theory for the branching ratio defined by Gutkowicz-Krusin *et al.* as

$$R(r_0) = \sum_{r=N}^{\infty} Q_\infty(r|r_0) \Big/ \sum_{r=-N}^{0} Q_\infty(r|r_0) \tag{13}$$

by noticing that the probability of absorption at $r \geqslant N$ is

$$\varphi(r_0) = R/(1 + R) \tag{14}$$

from which R can be found if $\varphi(r_0)$ is known. If the assumptions that permit the use of Eq. (5) are valid, then $\varphi(r_0)$ is known to satisfy[2]

$$\frac{\sigma^2}{2}\frac{d^2\varphi}{dr_0{}^2} + \mu_1 \frac{d\varphi}{dr_0} = 0 \tag{15}$$

subject to the boundary condition $\varphi(0) = 0$, $\varphi(N) = 1$. The consequences of this last equation are trivial and need not be given here.

There is one other general technique for finding the moments of occupation time and the absorption probabilities in one dimension. This is Wald's theorem,[3,4] which was developed for random walks that arise in sequential analysis. However, the use of Wald's theorem is intrinsically limited to one-dimensional problems. The diffusion limit analysis just presented for the one-dimensional case can be extended to deal with problems in higher dimensions, and will lead to useful results for sufficiently simple geometries.

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